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HW V , Math 530, Fall 2014

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QUESTION 1. (i) Let $(G, *)$ be a group.

a. Let $a \in G$ and let $C(a) = \{b \in G | a * b = b * a\}$. Prove that $C(a)$ is a subgroup of G.

b. Let $C(G) = \{b \in G | a * b = b * a$ for every $a \in G\}$. Prove that $C(G)$ is a normal subgroup of G. $C(G)$ is called the center of G, i.e., $C(G)$ is the set of all elements of G that commutate with every element in G.

c. Define a relation on G such that for every $a, b \in G$, $a \sim b$ iff $a = c * b * c^{-1}$ for some $c \in G$. Prove that $\sim$ is an equivalence relation on G.

d. Let $\sim$ be as above and let D be an equivalence class that contains an element $d \in G$. Prove that $|D| = 1$ if and only if $d \in C(G)$. Hence we can conclude that if $D_1, D_2, ..., D_n$ are all distinct equivalence classes such that $|D_i| = 1$ for every $1 \leq i \leq n$, then $D_1 \cup D_2 \cup \cdots \cup D_n = C(G)$. Thus, we conclude that if G is finite and $M_1, ..., M_k$ are all distinct equivalence classes where $|M_i| \geq 2$, then $|G| = |C(G)| + |M_1| + |M_2| + \cdots + |M_k|$.

e. Assume G is finite, let $\sim$ as above, and let M be an equivalence class such that $|M| \geq 2$. Suppose $b \in M$. Let F be the set of all distinct left cosets of $C(b)$. Prove that $|M| = |F|$. (hint: Define $H : F \to M$ such that $H(v * C(b)) = v * bv^{-1}$. Show that H is well-defined and bijection). Since $|F| = |G|/|C(b)|$, we conclude that $|M|$ is a factor of $|G|$.

f. Assume $|G| = p^m$ for some positive integer $m \geq 1$ and some prime integer p. Prove that $|C(G)| \geq p$. [Hint: Use the conclusion in (d) and (e)]

g. Assume $G/C(G)$ is cyclic. Prove that G is abelian.

h. Assume $|G| = p^2$ for some prime integer p. Prove that G is abelian.

(ii) Let G be an abelian group of order pq for some distinct prime integers p, q. Suppose that G has a subgroup of order q, say H and a subgroup of order p, say K. Prove that G is cyclic.

(iii) Let $(G, *)$ be an abelian group of order pq for some distinct prime integers p, q. Suppose that G has a subgroup of order q. Prove that G is cyclic.

(iv) Let $x = (1, 2)$ and $y = (2, 3, 4) \in S_4$. Find xoy and yox. Find $|xoy|$ and $|yox|$.

(v) Prove that S_8 has subgroups of order 15, 12, and 10 but S_8 has no elements of order 9, 14.

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